Characterizing Simplicial Commutative Algebras with Vanishing André-Quillen Homology

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Abstract. The use of homological and homotopical devices, such as Tor and André-Quillen homology, have found substantial use in characterizing commutative algebras. The primary category setting has been differentially graded algebras and modules, but recently simplicial categories have also proved to be useful settings. In this paper, we take this point of view up a notch by extending some recent uses of homological algebra in characterizing Noetherian commutative algebras to characterizing simplicial commutative algebras having *finite Noetherian homotopy* through the use of simplicial homotopy theory. These characterizations involve extending the notions of locally complete intersections and locally Gorenstein algebras to the simplicial homotopy setting.

Overview

Following a program set forth by Grothendieck (see [**6**]), a major research effort has been underway to characterize ring homomorphisms $f : R \to S$ of Noetherian rings. Use of homological devices have been pivotal to making such characterizations. For example, in [**17**] D. Quillen, in the process of developing a homology of commutative algebras, conjectured that the higner vanishing of this homology of S over R, in the case f is essentially of finite type and of finite flat dimension, charactertizes f as a locally complete intersection homomorphism. Motivated by this conjecture, L. Avramov [**4**] defined locally complete intersection homomorphisms more generally and established their properties, including providing a proof of Quillen's conjecture by a tour de force use of differentially graded and simplicial techniques. This fit into a larger program of Avramov with various collaborators to fulfill Grothendieck's program for rings and their homomorphisms. See, for example, [**4, 5, 8**]. In particular, characterizations of homomorphisms to be locally regular, complete intersection, Grothendieck, and Cohen-Macaulay were achieved.

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Also motivated by Quillen's conjecture and a perspective of commutative algebra from a strictly simplicial viewpoint (see, for example, [**12, 14**]), a version of Quillen's conjecture was formulated and proved for simplicial commutative algebras with *finite Noetherian homotopy* [**19, 20**], relying on techniques developed by analogy from the homotopy of spaces for the hootopy of simplicial commutative algebras. In the process, the notion of *homotopy complete intersection* was formulated and shown to be characterized by the vanishing of higher André-Quillen homology. The drawback was that the extension was valid only when the π_0 has non-zero characteristic. Since such restrictions is not needed in the constant simplicial case, as the main result of [**4**] clearly implies, then there is a gap in a full characterization of locally complete intersections through purely simplicial techniques.

The aim of this paper is twofold. The first is to begin to describe a program which extends the notions of complete intersection, Gorenstein, and Cohen-Macaulay to simplicial commutative algebras having Noetherian homotopy. This will involve summarizing the results of [**19, 20**] and extending some of the notions in [**5, 8**]. In particular, we will define the notions of *homotopy Gorenstein* and *homotopy Cohen-Macaulay* algebras. The second aim of this paper will be to characterize, using these notions, simplicial algebras with finite Noetherian homotopy, finite flat dimensional homotopy, and finite André-Quillen homology free of conditions on the characteristic of π_0. Specifically, we will prove the following (see §2 for definitions):

Homology Characterization Theorem. *Let A be a simplicial commutative R-algebra (R Noetherian) having finite Noetherian homotopy and $\mathrm{fd}_R(\pi_*A) < \infty$. If $D_s(A|R; -) = 0$ for $s \gg 0$, as a functor of $\pi_0 A$-modules, then:*

(a) *A is locally homotopy Gorenstein;*
(b) *If $\mathrm{char}(\pi_0 A) \neq 0$ then A is a locally homotopy complete intersection;*
(c) *If A is locally regularly 2-degenerate then $R \to A$ is a locally complete intersection homomorphism; that is, for each $\wp \in \mathrm{Spec}(\pi_0 A)$ there is a factorization (2.1) such that $\ker(\eta')$ is generated by a regular sequence.*

This paper is organized as folowed: §1 reviews the simplicial model structure for simplicial commutative algebras, as well as the tensor, hom, and differential structures and their derived functors. We also review relationships between certain simplicial categories and differentially graded categories. §2 reviews the definition of homotopy complete intersections and introduces the notions of homotopy Gorenstein and homotopy Cohen-Macaulay algebras. We end with a proof of the Homology Characterization Theorem.

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1. Homotopy theory of simplicial commutative algebras and simplicial modules

1.1. Simplicial model category structures. Fix a commutative ring R with unit and let $\mathcal{A}$ denote the category of commutative R-algebras with unit. If $B \in \mathcal{A}$, let $\mathcal{A}_B$ denote the subcategory of objects (A, ϵ) with $\epsilon : A \to B$ an R-algebra map. Given any category $\mathcal{C}$, we let $s\mathcal{C}$ denote the category of simplicial objects over $\mathcal{C}$. Finally, for $A \in s\mathcal{A}$, let $\mathcal{M}_A$ denote the category of simplicial A-modules.

Let $f : A \to B$ be a map in $s\mathcal{A}$. Recall from [**15**] that f is defined to be a:

(1) *weak equivalence* provided f is a weak equivalence of simplicial groups;
(2) *fibration* provided f is a fibration of simplicial groups;
(3) *cofibration* if and only if f is a retract of an *almost free map*, i.e. a map $f' : A' \to B'$ such that f' makes B' free as an almost simplicial A'-algebra (that is, without d_0).

By [**15**, §II.3], this structure makes $s\mathcal{A}_S$ a closed model category for any fixed $S \in s\mathcal{A}$.

Fixing $A \in s\mathcal{A}$, let $f : K \to L$ be a map in $\mathcal{M}_A$. Following [**15, 18**], we will say that f is defined to be a:

(1) *weak equivalence* provided Nf is a weak equivalence of chain complexes;
(2) *fibration* provided Nf is a level-wise surjection in positive degrees;
(3) *cofibration* if and only if Nf is a level-wise monomorphism whose cokernel in each degree $k \geq 0$ is a projective N_kA-module.

Here $N : \mathcal{M}_A \to Mod_{NA}$ is the normalized chain functor. By [**15**, §II.3], this structure makes $\mathcal{M}_A$ into a closed model category. Furthermore, part of the main theorem of [**18**] states

Schwede-Shipley Theorem: *The Dold-Kan correspondence*

$$N : \mathcal{M}_A \Longleftrightarrow Mod^+_{NA} : K$$

is a Quillen equivalence of symmetric monoidal closed model categories.

1.2. Tensor and Tor-modules. For $A \in s\mathcal{A}$, level-wise tensor product gives a functor

$$\otimes_A : \mathcal{M}_A \times \mathcal{M}_A \to \mathcal{M}_A.$$

In turn, this tensor product induces the *derived tensor product*

$$\otimes^{\mathbf{L}}_R : \mathrm{Ho}(\mathcal{M}_A) \times \mathrm{Ho}(\mathcal{M}_A) \to \mathrm{Ho}(\mathcal{M}_A)$$

on the homotopy category, which is defined by

$$K \otimes^{\mathbf{L}}_A L := X \otimes_A Y,$$

where X and Y are cofibrant replacements for K and L in $\mathcal{M}_A$, respectively.

Note also that $\otimes_A$ induces a functor on $s\mathcal{A}$ which descends to a functor $\otimes^{\mathbf{L}}_A$ on $s\mathcal{A}_B$, for any fixed B.

Next, define the *Tor-modules* for $K, L \in \mathcal{M}_A$ to be

$$\mathrm{Tor}^A_s(K, L) := \pi_s(K \otimes^{\mathbf{L}}_A L), \quad s \geq 0.$$

For each such s, Tor_s^A is a $\pi_0 A$-module.

A key method for computing Tor_*^A is the following device found in [**15**, II.6].

Kunneth Spectral Sequence: *For $K, L \in \mathcal{M}_A$, there is a first quadrant spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}_*^{\pi_* A}(\pi_* K, \pi_* L) \Longrightarrow \mathrm{Tor}_*^A(K, L).$$

1.3. Ext-modules. As noted above, in [**18**] it is shown that the normalization functor induces a functor $N : \mathcal{M}_A \to Mod^+_{NA}$ which is an equivalence of categories. We will therefore use N to define the *Ext-modules* for $K, L \in \mathcal{M}_A$. First, if T is a (commutative) DG ring and U, V are DG T-modules, let

$$\mathrm{Ext}^s_T(U, V) := H_{-s}(\mathrm{Hom}_T(P_*, I_*))$$

where $P_* \to U$ is a DG projective replacement and $V \to I_*$ is a DG injective replacement for NL in $NA - Mod$. Here the *Hom-complex* is defined as in [**9**, §1]. We then define, for $K, L \in \mathcal{M}_A$,

$$\mathrm{Ext}^s_A(K, L) := \mathrm{Ext}^s_{NA}(NK, NL).$$

A basic property of Ext is then given by:

Lemma 1.1. [**9**, (1.8)] *If $A \xrightarrow{\sim} B$ is a weak equivalence of simplicial algebras augmented over a field ℓ, then*

$$\mathrm{Ext}^*_A(\ell, A) \cong \mathrm{Ext}^*_B(\ell, B).$$

Now assume that ℓ is a field of characteristic 0. Let $\Re_\ell$ denote the category of commutative ℓ-algebras over ℓ, henceforth refered to as *rational ℓ-algebras*. Let $s_+\Re_\ell$ and $ch_+\Re_\ell$ denote the category of connected simplicial rational ℓ-algebras and the category of connected differentially graded rational ℓ-algebras, respectively.

Quillen's Theorem. [**16**, p. 223] *Normalization induces a functor $N : s_+\Re_\ell \to ch_+\Re_\ell$ which is an equivalence of closed model categories.*

Our aim now is to prove the following variation of [**9**, (4.1)].

Proposition 1.2. *Let $\psi : \ell[x] \to T$ be a map of augmented DG rings over ℓ from a free commutative graded ℓ-algebra on one generator x, $|x| = n > 0$, and let M be a DG $\ell[x]$-module. Assume furthermore that*

1. *$H_0(T)$ is Noetherian and each $H_i(T)$ is a finitely generated $H_0(T)$-module for $i \in \mathbb{Z}$;*
2. *$H(M)$ is bounded above.*

Letting $F(\psi) := T \otimes^{\mathbf{L}}_{\ell[x]} \ell$, there is an isomorphism of graded ℓ-modules

$$\mathrm{Ext}^*_T(\ell, M \otimes^{\mathbf{L}}{}_{\ell[x]} T) \cong \mathrm{Ext}^*_{\ell[x]}(\ell, M) \otimes_\ell \mathrm{Ext}^*_{F(\psi)}(\ell, F(\psi)).$$

Proof. By inspection of the statement of [**9**, 4.1], the key difference involves replacing an augmented DG ring R, for which $H_*(R) \cong H_0(R)$ is Noetherian, with $N(S_\ell(n))$. In their proof of (4.1), the condition on R insures that:

(1) There is a factorization
$$R \to X \xrightarrow{\simeq} \ell$$
with X a free R-algebra such that $X \xrightarrow{\simeq} X'$ with X' finite type over R with bounded below generators.

(2) There is a DG injective R-resolution $M \to I$ with I bounded above.

(1) and (2) insures that [**9**, (1.10)] can be applied to establish [**9**, (4.9)].

To show (1) in our context, let $\ell[x, y]$ be the free DG ℓ-algebra such that $|y| = n + 1$ and $\partial y = x$. Then there is a factorization
$$\ell[x] \to \ell[x, y] \xrightarrow{\simeq} \ell$$
with the required properties. Finally, (2) can be found in [**7**, (9.3.2.1)]. □

Let $S_\ell(n)$ be the free commutative ℓ-algebra generated by the Eilenberg-MacLane object $K(\ell, n)$. Let $A \in s\Re_\ell$ and let $\phi : S_\ell(n) \to A$ be a map in $s\Re_\ell$. ϕ determines a cofibration sequence in $\mathrm{Ho}(s\Re_\ell)$
$$S_\ell(n) \xrightarrow{\phi} A \to \mathcal{F}_\phi$$
where $\mathcal{F}_\phi := A \otimes^{\mathbf{L}}_{S_\ell(n)} \ell$.

Corollary 1.3. *If $A \in s_+\Re_\ell$ with π_*A a finite graded ℓ-module, then*
$$\mathrm{Ext}^*_A(\ell, A) \cong \mathrm{Ext}^*_{S_\ell(n)}(\ell, S_\ell(n)) \otimes_\ell \mathrm{Ext}^*_{\mathcal{F}_\phi}(\ell, \mathcal{F}_\phi).$$

Proof. Note first that $N(S_\ell(n)) \simeq \ell[x]$ with $|x| = n$. Since the result follows immediately from Proposition 1.2 for n odd ($\ell[x]$ is exterior on x), we assume that n is even. Using Quillen's Theorem, let $L(k) \in s\Re_\ell$ satisfy $NL(k) \simeq \ell[x]/(x^k)$. Thus, by Proposition 1.2:
$$\mathrm{Ext}^*_A(\ell, L(k) \otimes^{\mathbf{L}}{}_{S_\ell(n)} A) \cong \mathrm{Ext}^*_{S_\ell(n)}(\ell, L(k)) \otimes_\ell \mathrm{Ext}^*_{\mathcal{F}_\phi}(\ell, \mathcal{F}_\phi).$$
(Note that the Schwede-Shipley Theorem implies that $N\mathcal{F}_\phi \simeq F(N(\phi))$.)

Now $NS_\ell(n) \to NL(k)$ is equivalent to a map that is an isomomorphism in degrees $< nk$. By a Kunneth spectral sequence argument, since π_*A is bounded the map $A \to L(k) \otimes^{\mathbf{L}}{}_{S_\ell(n)} A$ induces a π_*-injection which is a π_*-isomorphism through degree nk, for $k \gg 0$. In particular, as π_*A-modules:
$$\mathrm{Tor}^{S_\ell(n)}_*(L(k), A) \cong \pi_*A \times \Sigma^{nk+1}\pi_*A.$$
Thus the map $A \to \lim_k(L(k) \otimes^{\mathbf{L}}{}_{S_\ell(n)} A)$ is a weak equivalence, whose normalization is equivalent to an isomorphism. Furthermore, using Quillen's Theorem and the Schwede-Shipley Theorem, the induced map $NA \to (\ell[x]/(x^k)) \otimes^{\mathbf{L}}{}_{\ell[x]} NA$ can be shown to be equivalent to a split injection, for $k \gg 0$, as NA-modules. Thus the result now follows from an argument using Lemma 1.1 and the Milnor sequence [**21**, §3.5]. □

Note: Corollary 1.3 holds over fields ℓ of arbitrary characteristic when $n = 1$.

1.4. Differentials and André-Quillen homology. Let A be a commutative ring and B a commutative A-algebra. If M is a B-module, recall that an A-module map $f : B \to M$ is a *derivation* provided $f(xy) = xf(y) + yf(x)$. Let $\mathrm{Der}_A(B, M)$ be the A-module of derivations. The functor $M \mapsto \mathrm{Der}_A(B, M)$ is representable: there is a canonically defined B-module $\Omega_{B|A}$, called the *differentials* of B over A, such that there is a natural isomorphism:

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega_{B|A}, M).$$

From the differentials, the *cotangent complex* of a simplicial commutative R-algebra A is defined by

$$\mathcal{L}(A|R) := \Omega_{X|R} \otimes_X A,$$

where $X \xrightarrow{\sim} A$ is a cofibrant replacement of A in $\mathrm{Ho}(s\mathcal{A})$. The *André-Quillen homology* of A over R with coefficients in the A-module M is then defined to be

$$D_*(A|R; M) := \pi_*(\mathcal{L}(A|R) \otimes_A M).$$

We now recall two important properties of André-Quillen homology:

(1) (Transitivity Sequence) Given maps $A \to B \to C$ in $s\mathcal{A}$ and a C-module M, there is a long exact sequence:

$$\ldots \to D_{s+1}(C|B; M) \to D_s(B|A; M) \to D_s(C|A; M) \to\to D_s(C|B; M) \to \ldots;$$

(2) (Flat Base Change) For $A, B \in s\mathcal{A}$ and M an $A \otimes^{\mathbf{L}}{}_R B$-module

$$D_*(A \otimes^{\mathbf{L}}{}_R B|B; M) \cong D_*(A|R; M).$$

A further useful relationship between homotopy and André-Quillen homology is given by the following:

Hurewicz Theorem: *For a connected simplicial R-algebra A over a field ℓ, there is a homomorphism*

$$h_* : \mathrm{Tor}_*^R(A, \ell) \to D_*(A|R; \ell)$$

for which h_* is an isomorphism in degrees $\leq n$ provided A is (n-1)-connected.

Let $\epsilon : A \to \ell$ be a commutative algebra over a field. Let $I = \ker(\epsilon)$. Define the *indecomposables* of A to be the ℓ-module $QA := I/I^2$. A well known result [**12, 14**] for A a supplemented ℓ-algebra is

$$\Omega_{A|\ell} \otimes_A \ell \cong QA.$$

We thus define the André-Quillen homology of a simplicial supplemented ℓ-algebra A by

$$H_*^Q(A) := \pi_* QX,$$

where $X \xrightarrow{\sim} A$ is a cofibrant resolution of A as simplicial supplemented algebras. Thus we have

$$H_*^Q(A) \cong D_*(A|\ell; \ell).$$

We now return to inspecting maps $\phi : S_\ell(n) \to A$ of simplicial supplemented ℓ-algebra. Our aim is to prove

PROPOSITION 1.4. *Let $\phi : S_\ell(n) \to A$ be a map of (n-1)-connected simplicial supplemented ℓ-algebras such that $H_n^Q(\phi)$ is an injection. If π_*A is a finite graded ℓ-module and $\pi_*(\mathcal{F}_\phi)$ is unbounded then $H_*^Q A$ is unbounded.*

(Recall that a positively graded module M is *unbounded* provided $M_t \neq 0$ for infinitely many t.)

To prove this proposition, we adapt the proof of [**10**, (4.2)]. Assume $\operatorname{char} \ell = 0$. Recall for a commutative DG ℓ-algebra T that a *minimal model* for T is a free DG-algebra $(\ell[X], \partial)$ such that $\partial X \subseteq I^2$, where I is the augmentation ideal of $\ell[X]$, together with a weak equivalence $\ell[X] \xrightarrow{\sim} T$. For existence and properties of minimal models, see [**11**]. Note that from Quillen's Theorem (see also [**17**, Thm. 9.5], if A is a simplicial supplemented ℓ-algebra and $\ell[X]$ is a minimal model for NA, then

$$H_*^Q(A) \cong QX.$$

Proof of Proposition 1.4. First, assume that $\operatorname{char} \ell \neq 0$. If π_*A and $H_*^Q A$ are both finite graded ℓ-modules, then, by the Algebraic Serre Theorem [**19**], $A \cong \bigotimes_I S_\ell(1)$ in the homotopy category. Thus $n = 1$ and $\mathcal{F}_\phi \cong \bigotimes_I S_\ell(1)$ with $|J| = |I| - 1$. It follows that $\pi_*\mathcal{F}_\phi$ is bounded.

Now assume that $\operatorname{char} \ell = 0$. Let $(\ell[X], \partial)$ be a minimal model for NA and let $\ell[x]$ be a minimal model for $N(S_\ell(n))$. By the assumption that $H_n^Q(\phi)$ is injective, it follows that we may assume that $x \in X$ and that the map $\ell[x] \to \ell[X]$ induced by the inclusion $\{x\} \subseteq X$ is equivalent to $N(\phi)$. It follows from Quillen's Theorem that if $X = \{x\} \bigcup Y$, then $\ell[Y]$ is a minimal model for $N\mathcal{F}_\phi$. Thus writing $\ell[X] \cong \ell[x] \otimes \ell[Y]$, we can express, for $u \in \ell[Y]$,

$$\partial(1 \otimes u) = 1 \otimes \bar{\partial} u + x \otimes \mathcal{O} u.$$

It follows that $\mathcal{O} : (\ell[Y], \bar{\partial}) \to (\ell[Y], \bar{\partial})$ is a derivation of degree -n-1.

Let J be the augmentation ideal of $\ell[Y]$. Let $\mathcal{O} = \Sigma_{i \geq 1} \mathcal{O}_i$ with $\mathcal{O}_i(J) \subseteq J^i$. Thus $\mathcal{O}_1 : J \to J$ is a derivation and, hence, induces

$$\mathcal{O}_1 : J/J^2 \to J/J^2.$$

Claim: *There exists an element u in the ℓ-dual $(J/J^2)^*$ such that $(\mathcal{O}_1^*)^n u \neq 0$ for all $n \geq 0$.*

It follows from this claim that $H_*^Q(\mathcal{F}_\phi)$ and, hence, $H_*^Q(A)$ are unbounded.

To establish the claim, assume for each $u \in (J/J^2)^*$ there is an $n \gg 1$ such that $(\mathcal{O}_1^*)^n u = 0$. Following the same argument as in [**10**, p.181], this implies that, for each $u \in J^*$, $(\mathcal{O}^*)^n u = 0$ for $n \gg 1$.

Now, consider the exact sequence

$$0 \to J \xrightarrow{\tau} I \to J \to 0,$$

where I is the augmentation ideal of $\ell[X]$ and $\tau(w) = z \otimes w$. Dualizing and applying cohomology, we obtain a long exact sequence

$$\ldots \to H^{i-1}(J^*) \xrightarrow{\delta} H^{i+n}(J^*) \to (H_{i+n}(I))^* \xrightarrow{\tau^*} H^i(J^*) \to \ldots$$

An easy computation shows that $\delta = H(\mathcal{O}^*)$. By our assumption on the finiteness of π_*A, $H_i(I) = 0$ for $i \geq N$, $N \gg 0$. Thus δ is injective for $i \geq N$. Since we are assuming $\pi_*\mathcal{F}_\phi$ is unbounded, this contradicts our local nilpotency condition on $(\mathcal{O})^*$. Thus our claim is established. $\square$

2. Characterizing Simplicial Commutative Algebras

We focus on extending the characterizations of homomorphisms $R \to S$ of Noetherian rings achieved in [**4, 5, 8**] to simplicial commutative R-algebras. To set in what direction this extension is to take, we view S as a constant simplicial R-algebra. To achieve a suitable type of extension, the notion of Noetherian needs to be spelled out for simplicial algebras. Such a notion was delineated and explored in [**20**], motivated by an analogous for concept for DG rings described in [**5**], which we now describe.

A simplicial commutative algebra A is said to have *Noetherian homotopy* provided

(1) $\pi_0 A$ is a Noetherian ring;
(2) each $\pi_m A$ is a finite $\pi_0 A$-module.

We furthermore say that A has *finite Noetherian homotopy* provided that (2) is replaced by

(2)′ $\pi_* A$ is a finite graded $\pi_0 A$-module.

The key to characterizing simplicial commutative R-algebras with Noetherian homotopy through homotopical/homological methods is to locally reduce to connected simplicial algebras over a field. Such objects yield more information under homological scrutinity. This approach was pioneered by L. Avramov [**3**] through the notion of DG fibre and used with great effect in [**5, 4**]. To adopt this approach in the simplicial setting, the following extension of the main theorem of [**8**] is needed

Factorization Theorem. [**20**, (2.8)] *Suppose A is a simplicial commutative R-algebra with Noetherian homotopy, R a Noetherian ring, and $\wp \in \mathrm{Spec}(\pi_0 A)$. Then there is a simplicial commutative algebra A' with Noetherian homotopy, such that $\pi_* A' \cong \widehat{\pi_* A}$, and there exists a (complete local) Noetherian R' that fits into the following commutative diagram in* $\mathrm{Ho}(s\mathcal{A}_{k(\wp)})$

$$\begin{array}{ccc} R & \xrightarrow{\eta} & A \\ \phi \downarrow & & \downarrow \psi \\ R' & \xrightarrow{\eta'} & A' \end{array} \tag{2.1}$$

with the following properties:

(1) *ϕ is a flat map and its closed fibre $R'/\wp R'$ is regular;*
(2) *ψ is a flat $D_*(-|R; k(\wp))$-isomomorphism;*
(3) *η' induces a surjection $\eta'_* : R' \to \pi_0 A'$;*

(4) $\operatorname{fd}_R(\pi_* A)$ *finite implies that* $\operatorname{fd}_{R'}(\pi_* A')$ *is finite*

We will call a choice of diagram (2.1) a *factorization* for a simplicial R-algebra A with Noetherian homotopy. Also, recall that, for an R-module M, $\operatorname{fd}_R M$ is the *flat dimension* of M.

We now describe extensions of the notions of locally complete intersection, locally Gorenstein, and locally CM (i.e. Cohen-Macaulay) for homomorphisms of Noetherian rings, as described in [**4, 5, 8**], to simplicial algebras with Noetherian homotopy.

To begin, let A be a connected simplicial supplemented ℓ-algebra, ℓ a field. We then declare that A is

(1) *a homotopy complete intersection* provided there is a finite set I with $A \cong \bigotimes_I S_\ell(1)$ in $\operatorname{Ho}(s\mathcal{A})$;
(2) *homotopy CM* provided there exists an $n \in \mathbb{Z}$ such that $\operatorname{Ext}^i_A(\ell, A) = 0$ for $i \neq n$;
(3) *homotopy Gorenstein* provided A is homotopy CM and $\dim_\ell \operatorname{Ext}^n_A(\ell, A) = 1$.

To get a sense of how these notions fit together and apply to basic examples of simplicial supplemented algebras, we prove the following:

Proposition 2.1. (a) *$S_\ell(n)$ is a homotopy complete intersection when $n = 1$ and homotopy Gorenstein in general when* $\operatorname{char} \ell = 0$.
(b) *We have the string of implications*

homotopy complete intersection $\Longrightarrow$ *homotopy Gorenstein* $\Longrightarrow$ *homotopy CM.*

Proof. To prove (a), we note that $NS_\ell(n) \simeq \ell[x]$, a free commutative graded algebra, on one generator x with $|x| = n$, and zero differential. Since $\ell[x]$ is Gorenstein it is therefore homotopy Gorenstein (see [**13**, (18.1) & (21.3)]). The result now follows from [**9**, (1.8)].

To prove (b), it is clearly enough to prove that a homotopy complete intersection is homotopy Gorenstein. But for this, note that an inclusion $\phi : S_\ell(1) \hookrightarrow \bigotimes_I S_\ell(1)$ onto one factor has homotopy fibre $\mathcal{F}_\phi \cong \bigotimes_J S_\ell(1)$ with $|J| = |I| - 1$. The result now follows from an induction using (a) and Proposition 1.3. □

Consider now a simplicial commutative R-algebra A over a field ℓ such that the unit map $R \to \pi_0 A$ is surjective. We will then say that A is a *homotopy complete intersection* (resp. *homotopy Gorenstein*, *homotopy CM*) *over* ℓ provided $A \otimes^{\mathbf{L}}{}_R\, \ell$ is a homotopy complete intersection (resp. homotopy Gorenstein, homotopy CM).

Finally, for a Noetherian ring R and a simplicial commutative R-algebra A with Noetherian homotopy, we say that A is a *locally homotopy complete intersection* (resp. *locally homotopy Gorenstein*, *locally homotopy CM*) provided that for each $\wp \in \operatorname{Spec}(\pi_0 A)$ there is a factorization (2.1) such that the simplicial R'-algebra A' is a homotopy complete intersection (resp. homotopy Gorenstein, homotopy CM) over the residue field $k(\wp)$.

Now we proceed to proving our main result. Before we do so, we need a technical definition and result. First, given a simplicial R-algebra A over a field ℓ, with (R, m)

local Noetherian and $\eta : R \to \pi_0 A$ a surjection, let $x_1, \ldots, x_n \in m$ be a maximal regular sequence in $\ker(\eta)$ which extends to a minimal generating set for $\ker(\eta)$. We then say that A is *regularly r-degenerate* provided the Kunneth spectral sequence

$$\mathrm{Tor}_*^{R/(x_1,\ldots,x_n)}(\pi_* A, \ell) \Longrightarrow \mathrm{Tor}_*^{R/(x_1,\ldots,x_n)}(A, \ell)$$

degenerates at the E^r-term. For a general simplicial commutative R-algebra A with Noetherian homotopy, we declare that A is *locally regularly r-degenerate* if for each $\wp \in \mathrm{Spec}(\pi_0 A)$, there is a factorization such that the simplicial R'-algebra A' is regularly r-degenerate over $k(\wp)$.

We will also need the following result.

Lemma 2.2. *For a simplicial R-algebra A over a field ℓ, let $x \in \ker(\eta)$ be a non-zero divisor in R. Then the sequence*

$$R/(x) \otimes^{\mathbf{L}}{}_R\, \ell \to A \otimes^{\mathbf{L}}{}_R\, \ell \to A \otimes^{\mathbf{L}}{}_{R/(x)}\, \ell$$

is a cofibration sequence of simplicial supplemented ℓ-algebras.

Proof. It is enough to check that

$$(A \otimes^{\mathbf{L}}{}_R\, \ell) \otimes^{\mathbf{L}}{}_{(R \otimes^{\mathbf{L}}{}_{R/(x)}\, \ell)}\, \ell \simeq A \otimes^{\mathbf{L}}{}_{R/(x)}\, \ell$$

but this is a straightforward computation. □

We now have reached the main goal of this paper.

Proof of the Homology Characterization Theorem. (a) For $\wp \in \mathrm{Spec}(\pi_0 A)$, choose a factorization (2.1) of A. Under the hypotheses on A, if $\mathrm{char}\, k(\wp) \neq 0$ then $A' \otimes^{\mathbf{L}}{}_{R'}\, \ell$ is a homotopy complete intersection, by Theorem B of [**20**]. Thus it is homotopy Gorenstein by Proposition 2.1 (b). So assume that $\mathrm{char}\, k(\wp) = 0$.

Since $X := A' \otimes^{\mathbf{L}}{}_{R'}\, \ell$ is connected, we may assume that X is (n-1)-connected for some $n \geq 1$. Let $\phi : S_\ell(n) \to X$ be a map so that $H_n^Q(\phi)(x)$ is a basis member of $H_n^Q(X) \cong D_n(A|R; k(\wp))$ (by Flat Base Change and the Factorization Theorem). Here we write $H_n^Q(S_\ell(n)) \cong \ell\langle x \rangle$.

Now, by Flat Base Change and the Transitivity Sequence,

$$H_s^Q(\mathcal{F}_\phi) \cong \begin{cases} H_s^Q(X) & s \neq n; \\ H_n^Q(X)/\ell\langle x \rangle & s = n. \end{cases}$$

Also, from the finite flat dimension condition and the Kunneth spectral sequence, $\pi_* X$ is a finite graded ℓ-module. It follows from Proposition 1.4 that $\pi_* \mathcal{F}_\phi$ is a finite graded ℓ-module. Thus, by an induction, using Proposition 2.1 (a), we may assume that $\mathcal{F}_\phi$ is homotopy Gorenstein. But now combining Proposition 2.1 (a) with Corollary 1.3 it follows that X is homotopy Gorenstein.

(b) This is just Theorem B of [**20**].

(c) We adapt another argument of L. Avramov and S. Halperin [**10**, (4.1)]. Again, choose a factorization (2.1) for A at $\wp \in \mathrm{Spec}(\pi_0 A)$. For the unit map $\eta' : (R', m') \to A'$, let $x_1, \ldots, x_n \in m'$ be a maximally regular sequence which extends to a minimal generating

set for $\ker(\eta')$. Note that if $x_1, \ldots, x_n$ fails to generate this kernel, then, by the main result of [**2**], $\mathrm{Tor}_*^{R'/(x_1,\ldots,x_n)}(\pi_* A', k(\wp))$ is unbounded. Since we are assuming A is locally regularly 2-degenerate, we may assume that $\mathrm{Tor}_*^{R'/(x_1,\ldots,x_n)}(A', k(\wp))$ is unbounded.

Now, for $x \in \ker(\eta')$ a non-zero divisor in R', we have

$$H_*^Q(R'/(x) \otimes^{\mathbf{L}}{}_{R'} \ell) \cong D_*(R'/(x)|R'; k(\wp)) \cong k(\wp)\langle u|\ |u| = 1\rangle,$$

by [**1**, (6.25)]. Thus $R'/(x) \otimes^{\mathbf{L}}{}_{R'} k(\wp) \cong S_{k(\wp)}(1)$ in the homotopy category, by [**19**, (2.1.3)]. Thus, by an induction, using Lemma 2.2, there is a cofibration sequence

$$S_\ell(1) \to A' \otimes^{\mathbf{L}}{}_{(R'/(x_1,\ldots,x_{s-1}))} k(\wp) \to A' \otimes^{\mathbf{L}}{}_{(R'/(x_1,\ldots,x_s))} k(\wp)$$

for each $1 \leq s \leq n$. By the Factorization Theorem and our discussion above, there exists an $1 \leq s \leq n$ such that

(1) $\mathrm{Tor}_*^{R'/(x_1,\ldots,x_{s-1})}(A, k(\wp))$ is bounded;
(2) $\mathrm{Tor}_*^{R'/(x_1,\ldots,x_s)}(A, k(\wp))$ is unbounded.

Furthermore, by Flat Base Change, the Transitivity Sequence, and our finiteness assumptions, we also have

(3) $D_*(A'|R'/(x_1,\ldots,x_t); k(\wp))$ is bounded for all $1 \leq t \leq n$.

Applying Proposition 1.4, (1) and (2) together imply that $D_*(A'|R'/(x_1,\ldots,x_{s-1}); k(\wp))$ is unbounded, contradicting (3). We therefore conclude that $\ker(\eta') = (x_1, \ldots, x_n)$. $\square$

Remark: Proposition 2.1 and the Homology Characterization Theorem shows that the standard stratified characterization of Noetherian rings

$$\text{Complete Intersection} \subseteq \text{Gorenstein}$$

extends to our present homotopy setting, yet sensitivity to differences in characteristic appear. Nevertheless, we now begin to fill a gap between the main result of [**4**], which is independent of characteristic, and the main results of [**19, 20**], which are valid only in non-zero characteristics. In particular, we have the following consequence of part (c) of this theorem.

Corollary *Let $\phi : R \to S$ be a homomorphism of Noetherian rings with $\mathrm{fd}_R S < \infty$. Then $D_s(S|R; -) = 0$ for $s \gg 0$ implies that ϕ is a locally complete intersection homomorphism.*

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